MATH 201: Calculus and Analytic Geometry III

Fall 2017-2018, Exam 2, Duration: 60 min.

| Problem | 1a | 1b | 1c | 2 | 3 | 4a | 4b | 5a | 5b | 6 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Points | 7 | 7 | 7 | 10 | 20 | 10 | 10 | 10 | 10 | 9 | 100 |
| Scores | 7 | 7 | 7 | 10 | 20 | 10 | 10 | 10 | 9 | 9 | 99 |

**Name:**

**AUB ID:**

insight club

Please circle your section:

| | | | |
|---|---|---|---|
| Section 1<br>MWF 3, Karam<br>Recitation F. 11 | Section 2<br>MWF 3, Karam<br>Recitation F. 8 | Section 3<br>MWF 3, Karam<br>Recitation F. 10 | Section 4<br>MWF 3, Karam<br>Recitation F. 9 |
| Section 5<br>MWF 10, Shayya<br>Recitation T. 11 | Section 6<br>MWF 10, Shayya<br>Recitation T. 12:30 | Section 7<br>MWF 10, Shayya<br>Recitation T. 2 | Section 8<br>MWF 10, Shayya<br>Recitation T. 5 |
| Section 9<br>MWF 11, Yamani<br>Recitation F. 3 | Section 10<br>MWF 11, Yamani<br>Recitation F. 4 | | |
| Section 12<br>MWF 2, Nahlus<br>Recitation M. 8 | Section 13<br>MWF 2, Nahlus<br>Recitation M. 9 | | |
| Section 16<br>MWF 9, Mourtada<br>Recitation Th. 9:30 | Section 17<br>MWF 9, Mourtada<br>Recitation Th. 2 | | |
| Section 19<br>MWF 1, Nahlus<br>Recitation F. 10 | Section 20<br>MWF 1, Nahlus<br>Recitation F. 8 | | |
| Section 22<br>MWF 10, Abi-Khuzam<br>Recitation F. 4 | Section 23<br>MWF 10, Abi-Khuzam<br>Recitation F. 2 | | |
| Section 26<br>MWF 11, Aoun<br>Recitation Th. 11 | Section 27<br>MWF 11, Aoun<br>Recitation Th. 12:30 | Section 28<br>MWF 11, Aoun<br>Recitation Th. 5 | |

**Notes before solving the exam:**

1) You have to solve the recommended problems in the book after understanding each chapter from the book and the notes.

2) Please understand that this exam is solved by students, and it may contain some mistakes.

3) If you have any questions or concerns, let us know through our mail: insightclub@gmail.com.

GOOD LUCK :)

INSTRUCTIONS:

(a) Explain your answers precisely and clearly to ensure full credit.

(b) Closed book. No notes. No calculators. No cellphones.

(c) UNLESS CLEARLY SPECIFIED OTHERWISE, THE BACKSIDE OF THE PAGES WILL NOT BE GRADED,

**Problem 1**

(7 pts each) Determine if the limit of each of the following functions exists as $(x, y) \to (0,0)$.

Explain

**(a)** $f(x,y) = \dfrac{y^4}{x^2 + y^2}$



since $x^2 + y^2 \geq 0$ (denominator is +)

$$0 \leq \frac{y^4}{x^2+y^2} \leq \frac{y^4}{y^2} = y^2$$



but $\lim\limits_{\substack{x \to 0 \\ y \to 0}} y^2 = 0$ then by C.S.T

$$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^2+y^2} = 0$$ then limit exist

and is equal to $0$



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**(b)** $g(x,y)=\dfrac{x^2}{|x|+9y^4}$

since $|x|+9y^4 \geq 0$ (denominator is +)

$$0 \leq \frac{x^2}{|x|+9y^4} \leq \frac{x^2}{|x|} = |x|$$

but $\lim\limits_{x\to 0} |x| = 0$ then by C.S.T

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{|x|+9y^4} = 0$$ then limit exists

and is equal to 0

**(c)** $h(x,y) = \dfrac{xy}{3y-4x}$

let $3y - 4x = m x^{\alpha+1}$

we have:

then $y = \dfrac{m x^{\alpha+1} + 4x}{3}$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{3y-4x} = \lim_{x\to 0} \frac{x\left(\frac{4}{3}x + \frac{m}{3}x^{\alpha+1}\right)}{m x^{\alpha+1}}$$

$$= \lim_{x\to 0} \frac{x^2\left(\frac{4}{3} + \frac{m}{3}x^{\alpha}\right)}{m x^{\alpha+1}}$$

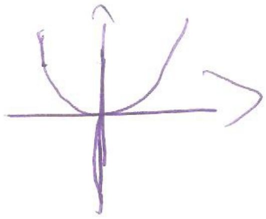
- for $\alpha = 1$ we get:

$$\lim_{x\to 0} \frac{\cancel{x^2}\left(\frac{4}{3} + \frac{m}{3}x\right)}{\cancel{x^2}\, m} = \frac{4}{3m}$$

since the limit is in terms of $m$, then the limit has different values for different values of $m$

$\Rightarrow$ lim doesn't exist

**Problem 2**



(10 pts) Consider the function $f(x,y)=\dfrac{x}{\sqrt{y-x^2}}$

Find the domain of the function $f$. Decide if the domain of $f$ is an open region, a closed region, or neither. Also decide if Domain $f$ is bounded or unbounded.

Domain of f: ~~$\mathbb{R}^2$~~ all pts $(x,y)$ in $\mathbb{R}^2$ ~~except for~~
such that $y-x^2>0$

- ~~Domain is a parabola of equation in~~
- Domain is open since all its pts are interior pts
- Domain is not closed since it doesn't include its boundary pts
- Domain is not bounded since it can't be placed in a ~~c~~ disc of fixed radius.
- Domain is ~~inside~~ interior of the parabola of eq: $y-x^2=0$

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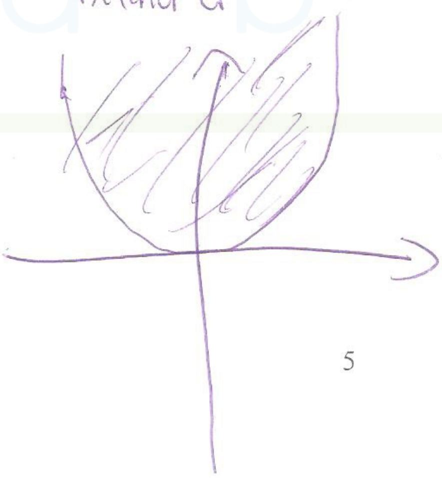

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**Problem 3**

(20 pts) Find the tangent plane and normal line to the surface $2x^2 + 4e^y = 6 - 3\ln z$ at the point $(1,0,1)$.

~~$2x^2 + 4e^y - 6 + 3\ln$~~

$2x^2 + 4e^y - 6 + 3\ln z = 0$

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let $f(x,y,z) = 2x^2 + 4e^y - 6 + 3\ln z$

$f_x = 4x$ at $P(1,0,1)$: $f_x = 4$

$f_y = 4e^y$ at $P(1,0,1)$: $f_y = 4$

$f_z = \frac{3}{z}$ at $P(1,0,1)$: $f_z = 3$

eq. of tg plane: $f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$4(x-1) + 4(y-0) + 3(z-1) = 0$

$4x + 4y + 3z = 7$ is the eq. of tg plane at $P(1,0,1)$

since $P(1,0,1)$ is a pt on the plane and $\vec{\nabla f}$ is a normal vector of the plane eq. of normal line: (at $P(1,0,1)$)

$$\left\{ \begin{array}{l} \cancel{x = 3m} \quad x = 4m + 1 \\ y = 4m \\ z = 3m + 1 \end{array} \right. \qquad \boxed{m \in \mathbb{R}}$$

since the normal line has a direction vector same as $\vec{\nabla f}$

and has a point $P(1,0,1) \in$ normal line

**Problem 4**

Let $f(x,y,z)$ be a differentiable function of three variables. Suppose that

$\nabla f(1,1,2) = 6\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k} \qquad \nabla f(6,2,4) = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$

Let

$x = r^2 + 2s$

$y = \dfrac{r}{s}$

$z = 2r + \ln s$

and $w(r,s) = f(x,y,z)$.

**(a)** (10 pts) Find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ at the point $(r,s) = (2,1)$.

The following ~~pub~~ $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ are solved using $\boxed{\text{chain rule}}$. for $(r,s) = (2,1)$ we get:

$x = 6 \quad y = 2 \quad z = 4$ and $\nabla f(6,2,4) = \vec{i} + \vec{j} + \vec{k}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= (1)(2(2)) + (1)\left(\frac{1}{1}\right) + (1)(2) = 7$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= (1)(2) + (1)\left(\frac{-2}{1}\right) + (1)(1) = 1$$

**(b)** (10 pts) Find the minimum value and maximum value of the directional derivative of $f(x,y,z)$ at the point $(1,1,2)$.

at pt $(1,1,2)$ $\quad \nabla f(1,1,2) = 6\vec{i} - 2\vec{j} + \vec{k}$

$D_u f = \vec{\nabla} f \cdot \vec{u} = \|\nabla f\| \cdot \|\vec{u}\| \cdot \cos(\theta) = \|\vec{\nabla} f\| \cdot \cos(\theta)$ (since $\vec{u}$ is a unit vector)

$D_u f$ is max when it is increasing most rapidly then:
($D_u f$ is max for $\theta = 0$ then $\cos(\theta)$ is max $\cos(0) = 1$)

$$D_u f_{max} = \|\nabla f\| = \sqrt{36+4+1} = \sqrt{41}$$

$D_u f$ is min. when it is decreasing most rapidly then:

$$D_u f_{min} = -\|\nabla f\| = -\sqrt{36+4+1} = -\sqrt{41}$$

($D_u f$ is minimum when $\cos(\theta)$ is minimum which is for $\theta = \pi$)
$\cos(\pi) = -1$

**Problem 5**

**(a)** (10 pts) Find the Taylor polynomial $p_1(x)$ generated by the function $f(x)=\sqrt[3]{1+x}$ at the center $a=0$.



$f(x) = \sqrt[3]{1+x} = (x+1)^{1/3}$ ; $f(0) = (1)^{1/3} = 1$

$f'(x) = \frac{1}{3}(x+1)^{-2/3}$ ; $f'(0) = \frac{1}{3}(1)^{-2/3} = \frac{1}{3}$

$P_1(x) = \frac{f(0)}{(0)!} + \frac{f'(0)}{(1)!}x$

then:



$$\boxed{P_1(x) = 1 + \frac{x}{3}}$$

which is the Taylor polynomial at $a=0$

**(b)** (10 pts) Use Taylor's theorem to estimate the error resulting from the approximation

$$f(x) \approx p_1(x) \qquad \text{for } 0 \le x \le 0.3$$

(Do not simplify your answer. Leave your answer as a fraction.)



$f(x) = P_1(x) + R_1(x) \approx P_1(x)$



but $R_1(x) = \dfrac{f''(c)\, x^2}{2!}$ where $c$ is between $0$ and $x$

$f''(x) = \dfrac{-2}{9}(x+1)^{-5/3}$ then $f''(c) = \dfrac{-2}{9}(c+1)^{-5/3}$

then $R_1(x) = \dfrac{\left(\frac{-2}{9}\right)(c+1)^{-5/3}(x^2)}{2!} = \left(\dfrac{-1}{9}\right)\left(\dfrac{1}{\sqrt[3]{(c+1)^5}}\right) x^2$

$\leq \left(\dfrac{-1}{9}\right)\left(\dfrac{1}{\sqrt[3]{(1)^5}}\right)(0.3)^2$

$-1$

**Problem 6**

(9 pts) Let $f(x)$ be an infinitely differentiable function of one variable. Suppose that

- $f(5)=1$
- $f^{(n)}(5)=1$ for $n=1, 2, \ldots$
- $\lim_{n\to\infty} R_n(x)=0$ for all $x$, where $R_n(x)$ is the remainder in Taylor's theorem at the center $a=5$.

Find the exact value of $f(10)$.





at $a=5$: $f(x) = P_n(x) + R_n(x)$

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-5)^n f^n(5)}{n!} + \cancel{R_n(x)} \sum_{n=0}^{\infty} \frac{(x-5)^{n+1} f^{n+1}(5)}{(n+1)!}$$

but $\lim_{n\to\infty} R_n(x) = 0$ then:



$$f(x) = 1 + (x-5) + \frac{(x-5)^2}{2!} + \frac{(x-5)^3}{3!} + \cdots$$

for $x=5$ $f(5)=1$



$$f(10) = 1 + (10-5) + \frac{(10-5)^2}{2!} + \frac{(10-5)^3}{3!} + \cdots + \frac{(10-5)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{5^n}{n!} = e^5$$